Polyhedral Nonexistence from Graphs

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Abstract

For extra credit for Joseph S.B. Mitchell's Computational Geometry class at Stony Brook University, I developed a short proof for the nonexistence of a genus 0 polyhedron with 7 faces. This paper presents that proof openly and offers some natural extensions.

1 Introduction

This paper starts by showing the nonexistence of a particular type of genus 0 polyhedron, one with 7 faces, all of which are 4-gons. The original proof was a one page proof as an extra credit excercise. Upon revisiting it as a TA for the course, I came up with some generalizations. This paper will cover those generalizations, flesh out a step in the proof I think needs a bit more justification, and pose some open questions.

2 The Short Proof

Theorem 2.1. *There is no genus 0 polyhedron that has exactly 7 faces, each being a 4-gon.*

Proof. Assume that there exists a polyhedron P with exactly 7 faces, each being a 4-gon.
Since every face is a 4-gon, we know that each face has 4 edges and that each of these edges is shared by two faces (to avoid double counting). From these two properties we can deduce the number of edges:

$$E = \frac{4F}{2} = \frac{4(7)}{2} = 14$$

Using the Euler characteristic, which is equal to 2 for genus 0 shapes (Originally from Euler [2], also Theorem 6.12 in Devadoss and O'Rourke [1]), we can determine the number of vertices

$$\begin{aligned}
\chi(P) &= 2 \\
V - E + F &= 2 \\
V - 14 + 7 &= 2 \\
V &= 9
\end{aligned}$$

Since every face is a 4-gon, every circuit in the graph of its vertices must be of even length (if not obvious, shown later).
Therefore, the graph of its vertices must be bipartite. (Theorem 1.3.2 in Tucker's Applied Combinatorics [3])
If the graph has 9 vertices and is bipartite, the larger of the two sets in the partition must have at least 5 vertices.
Since every vertex in a polyhedron must have at least degree 3 and there can't be any edges between the elements of the partitioned sets, there must be at least 3 edges for every vertex in the larger of the partitioned sets.

$$E \geq 3 \cdot 5 \geq 15$$

Which contradicts our earlier calculation that the edge count is 14.
Therefore, there is no genus 0 polyhedron that has exactly 7 faces, each being a convex 4-gon. ■

3 Justifying even cycles

In the short proof, a key aspect is the bipartiteness of polyhedrons with 4-gon faces. This rests on the fact that polyhedrons with 4-gon faces have only even cycles. To show this, I will generalize in the following theorem.

Theorem 3.1. *The corresponding graph formed from the vertices and edges of a polyhedron with faces that all have even side counts must contain only even cycles.*

Proof. We show this by induction on the induced subgraphs from the polyhedron.

Base Case: Only vertex is included. There are no cycles, so by default, all of them are even.

Inductive step: There are two cases, either the vertex being added does not complete any faces or it completes some number of faces. The first case is trivial, since no faces are completed, no new cycles are completed, so the induction holds.

Our second case entails using an argument from the fact the induced graph so far has been bipartite. We know this from the inductive hypothesis. Every vertex that is adjacent to the newly added vertex must have a different color than it will have. If the graph so far has been connected, all the vertices must have the same color or an odd-face will have been formed. An odd-face cannot be formed at any point in the construction since any attempt at splitting it will either produce an odd an even face or require the creation of a third face which will have an odd number of sides.

If the graph is not connected, then adjacent vertices from the same component will have the same color, but the colors may be different from different components. If this is the case then simply invert the colorings of the smaller set of components.

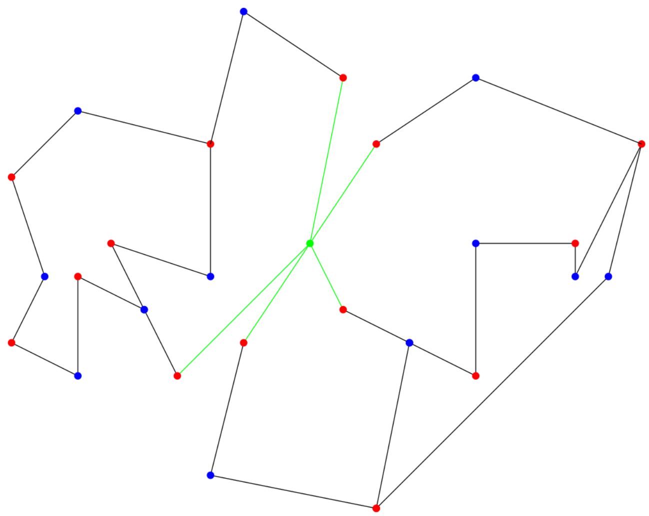

Figure 1: Face completion preserves bipartiteness

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4 Generalized Genuses

Theorem 4.1. *There is no polyhedron that has exactly 7 faces, each being a 4-gon with genus greater than 0.*

Proof. We can extend the proof from the first section to higher genuses using contradiction again. First we evaluate the number of vertices and edges from Euler characteristic [2], with accounting for genus.

$$V - E + F = 2 - 2g$$

Since our edge and face counts ($E = 14, F = 7$) from earlier have nothing to do with genus, we can keep those and solve for the number of vertices.

$$V = 9 - 2g \leq 7$$

From here, we use the opposite logic, try to show that if genus is greater than 0, we can't possibly have enough edges because it is bipartite. Edges can only go from one of the partitions to the other by definition, so the total number of edges is at most the cardinality of one set times the cardinality of the other. Let R be the cardinality of one of the sets.

$$E \leq (7 - R)R = 7R - 7R^2$$

R must be an integer, but even if extended to reals, E can have a maximum value of 12.25 at $R = 3.5$ from differentiating with respect to R and setting to 0.

$$E \leq 12.25$$

Which is a contradiction, since $E = 14$.

■

5 Generalized Faces

Theorem 5.1. *If a genus 0 polyhedron with $2n + 1$ faces, all with $2m$ sides exists,* $\frac{-8n+3}{(4m-4)(n+1)+10} - m + n + \frac{1}{2} \leq mn \leq 3n + m - 3\lceil (m+1)/2 \rceil$.

Proof. First we find the number of faces and edges.

$$F = 2n + 1$$

$$E = \frac{2m(2n+1)}{2} = 2mn + m$$

Now we will solve for vertices using Euler Characterstic [2].

$$\begin{aligned} \chi(P) &= 2 \\ V - E + F &= 2 \\ V - 2mn - m + 2n + 1 &= 2 \\ V &= 2mn + m - 2n + 1 \end{aligned}$$

Every vertex must have 3 edges and from earlier we know that the graph of this polyhedron would be bipartite, so the number of edges must be at least 3 times the cardinality of the larger of the two sets of the bipartition.

$$E \geq 3 \cdot \lceil V/2 \rceil \geq 3 \cdot \lceil mn + m/2 - n + 1/2 \rceil \geq 3 \cdot (mn - n + \lceil (m+1)/2 \rceil$$

$$2mn + m \geq 3mn - 3n + 3\lceil (m+1)/2 \rceil$$

$$mn \leq 3n + m - 3\lceil (m+1)/2 \rceil$$

Now we have one side of the inequality, we derive the other using the same argument as for the higher genuses in the specific case. Once again, let R be the number of vertices in one of the sets in the bipartition.

$$E \leq VR - R^2$$

We differentiate and solve for 0 to find the maximum R.

$$0 = V - 2R$$

$$R = V/2$$

$$VR - R^2 = V^2/2 - V^2/4 = V^2/4$$

$$E \leq (2mn + 2m - 2n + 3)^2/4$$

$$2mn + 2m \leq (2mn + 2m - 2n + 3)^2/4$$

$$-4(2n+3) \leq (2mn + 2m - 2n + 3)^2 - 4(2mn + 2m) - 4(2n+3)$$

$$-4(2n+3) \leq (2mn + 2m - 2n + 3 - 4)(2mn + 2m - 2n + 3)$$

$$\frac{-8n+3}{2mn + 2m - 2n + 3} \leq 2mn + 2m - 2n - 1$$

$$mn \geq \frac{-8n+3}{4mn + 4m - 4n + 6} - m + n + \frac{1}{2}$$

$$mn \geq \frac{-8n+3}{4m(n+1) - 4n + 6} - m + n + \frac{1}{2}$$

$$mn \geq \frac{-8n+3}{(4m-4)(n+1) + 10} - m + n + \frac{1}{2}$$

■

6 Open Questions

A cleaner conclusion for the general case is desirable, as the inequality relating the product of m and n does not seem to really give too much detail on the choices of 2m and 2n+1. Also allowing for multiple types of even faces might make things interesting. It also opens up interesting questions computationally with regards to the graphs induced by polyhedra, if we can define a problem on a set of polyhedrons that are characterized by having an odd number of even sided faces, we can potentially develop algorithms (or hardness proofs) based on the constraints put forth here.

References

[1] Satyan L. Devadoss and Joseph O'Rourke. *Discrete and Computational Geometry.* Princeton University Press, 2011. URL: http://press.princeton.edu/titles/9489.html.

[2] Leonhard Euler. *Elementa doctrinae solidorum.* 1758.

[3] Alan Tucker. *Applied Combinatorics.* John Wiley Sons, Inc., USA, 2006.